

# A long pseudo-comparison of premice in $L[x]$

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## Abstract

We describe an obstacle to the analysis of  $\text{HOD}^{L[x]}$  as a core model: Assuming sufficient large cardinals, for a Turing cone of reals  $x$  there are premice  $M, N$  in  $\text{HC}^{L[x]}$  such that the pseudo-comparison of  $L[M]$  with  $L[N]$  succeeds, is computed in  $L[x]$ , and lasts through  $\omega_1^{L[x]}$  stages. Moreover, we can take  $M = M_1|(\delta^+)^{M_1}$  where  $M_1$  is the minimal iterable proper class inner model with a Woodin cardinal, and  $\delta$  is that Woodin. We can take  $N$  such that  $L[N]$  is  $M_1$ -like and short-tree-iterable.

## 1 Introduction

A central program in descriptive inner model theory is the analysis of  $\text{HOD}^W$ , for transitive models  $W$  satisfying  $\text{ZF} + \text{AD}^+$ ; see [6], [5], [7], [4]. For the models  $W$  for which it has been successful, the analysis yields a wealth of information regarding  $\text{HOD}^W$  (including that it is fine structural and satisfies GCH), and in turn about  $W$ .

Assume that there are  $\omega$  many Woodin cardinals with a measurable above. A primary example of the previous paragraph is the analysis of  $\text{HOD}^{L(\mathbb{R})}$ . Work of Steel and Woodin showed that  $\text{HOD}^{L(\mathbb{R})}$  is an iterate of  $M_\omega$  augmented with a fragment of its iteration strategy (where  $M_n$  is the minimal iterable proper class inner model with  $n$  Woodin cardinals). The addition of the iteration strategy does not add reals, and so the  $\text{OD}^{L(\mathbb{R})}$  reals are just  $\mathbb{R} \cap M_\omega$ . The latter has an analogue for  $L[x]$ , which has been known for some time: for a cone of reals  $x$ , the  $\text{OD}^{L[x]}$  reals are just  $\mathbb{R} \cap M_1$ . Given this, and further analogies between  $L(\mathbb{R})$  and  $L[x]$  and their respective HODs, it is natural to ask whether there the full  $\text{HOD}^{L[x]}$  is an iterate of  $M_1$ , adjoined with a fragment of its iteration strategy. Woodin has

conjectured that this is so for a cone of reals  $x$ ; for a precise statement see [2, 8.23]. Woodin has proved approximations to this conjecture. He analyzed  $\text{HOD}^{L[x,G]}$ , for a cone of reals  $x$ , and  $G \subseteq \text{Coll}(\omega, < \kappa)$  a generic filter over  $L[x]$ , where  $\kappa$  is the least inaccessible of  $L[x]$ ; see [2, 8.21] and [7]. However, the conjecture regarding  $\text{HOD}^{L[x]}$  is still open.

In this note, we describe a significant obstacle to the analysis of  $\text{HOD}^{L[x]}$ .

Before proceeding, we give a brief summary of some relevant definitions and facts. We assume familiarity with the fundamentals of inner model theory; see [6], [3]. One does not really need to know the analysis of  $\text{HOD}^{L[x,G]}$ , but familiarity does help in terms of motivation; the system  $\mathcal{F}$  described below relates to that analysis. We do rely on some smaller facts from [7, §3]. Let us give some terminology, and recall some facts from [7]. We say that a premouse  $N$  is *pre- $M_1$ -like* iff  $N$  is proper class, 1-small, and has a (unique) Woodin cardinal, denoted  $\delta^N$ . (The notion  *$M_1$ -like* of [7] is stronger; it has some iterability built in.) Let  $P, Q$  be pre- $M_1$ -like. Given a normal iteration tree  $\mathcal{T}$  on  $P$ ,  $\mathcal{T}$  is *maximal* iff  $\text{lh}(\mathcal{T})$  is a limit and  $L[M(\mathcal{T})]$  has no  $\mathcal{Q}$ -structure for  $M(\mathcal{T})$  (so  $L[M(\mathcal{T})]$  is pre- $M_1$ -like with Woodin  $\delta(\mathcal{T})$ ). A premouse  $R$  is a (*non-dropping*) *pseudo-normal iterate* of  $P$  iff there is a normal tree  $\mathcal{T}$  on  $P$  such that either  $\mathcal{T}$  has successor length and  $R = M_\infty^\mathcal{T}$ , the last model of  $\mathcal{T}$  (and  $[0, \infty]_\mathcal{T}$  does not drop), or  $\mathcal{T}$  is maximal and  $R = L[M(\mathcal{T})]$ . A *pseudo-comparison* of  $(P, Q)$  is a pair  $(\mathcal{T}, \mathcal{U})$  of normal iteration trees formed according to the usual rules of comparison, such that either  $(\mathcal{T}, \mathcal{U})$  is a successful comparison, or either  $\mathcal{T}$  or  $\mathcal{U}$  is maximal. A (*z*-)*pseudo-genericity iteration* is defined similarly, formed according to the rules for genericity iterations making a real (*z*) generic for Woodin's extender algebra. We say that  $P$  is *normally short-tree-iterable* iff for every normal, non-maximal iteration tree  $\mathcal{T}$  on  $P$  of limit length, there is a  $\mathcal{T}$ -cofinal wellfounded branch through  $\mathcal{T}$ , and every putative normal tree  $\mathcal{T}$  on  $P$  of length  $\alpha + 2$  has wellfounded last model (that is, we never encounter an illfounded model at a successor stage). If  $P \restriction \delta^P \in \text{HC}^{L[x]}$ , then normal short-tree-iterability is absolute between  $L[x]$  and  $V$ . If  $P, Q$  are normally short-tree-iterable then there is a pseudo-comparison  $(\mathcal{T}, \mathcal{U})$  of  $(P, Q)$ , and if  $\mathcal{T}$  has a last model then  $[0, \infty]_\mathcal{T}$  does not drop, and likewise for  $\mathcal{U}$ .

It has been suggested<sup>1</sup> that one might analyze  $\text{HOD}^{L[x]}$  using an  $\text{OD}^{L[x]}$  directed system  $\mathcal{F}$  such that:

- the nodes of  $\mathcal{F}$  are pairs  $(N, s)$  such that  $s \in \text{OR}^{<\omega}$  and  $N$  is a normally

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<sup>1</sup>For example, at the AIM Workshop on Descriptive inner model theory, June, 2014.

short-tree iterable, pre- $M_1$ -like premouse with  $N|\delta^N \in \text{HC}^{L[x]}$  and such that there is an  $L[N]$ -generic filter  $G$  for  $\text{Coll}(\omega, \delta^N)$  in  $L[x]$ ,<sup>2</sup>

- for  $(P, t), (Q, u) \in \mathcal{F}$ , we have  $(P, t) \leq_{\mathcal{F}} (Q, u)$  iff  $t \subseteq u$  and  $Q$  is a pseudo-iterate of  $P$ , and
- $(M_1, \emptyset) \in \mathcal{F}$ .

There are also further conditions, regarding the sets  $s$ , strengthening the iterability requirements; these and other details regarding how the direct limit is formed from  $\mathcal{F}$  are not relevant here.

The main difficulty in analyzing  $\text{HOD}^{L[x]}$  in this manner is in arranging that  $\mathcal{F}$  be directed. For this, it seems most obvious to try to arrange that  $\mathcal{F}$  be closed under pseudo-comparison of pairs.

However, we show here that, given sufficient large cardinals, there is a cone of reals  $x$  such that if  $\mathcal{F}$  is as above, then  $\mathcal{F}$  is *not* closed under pseudo-comparison. The proof proceeds by finding a node  $(N, \emptyset) \in \mathcal{F}$  such that, letting  $(\mathcal{T}, \mathcal{U})$  be the pseudo-comparison of  $(M_1, N)$ , then  $\mathcal{T}, \mathcal{U}$  are in fact pseudo-genericity iterations of  $M_1, N$  respectively, making reals  $y, z$  generic, where  $\omega_1^{L[y]} = \omega_1^{L[z]} = \omega_1^{L[x]}$ . Letting  $W$  be the output of the pseudo-comparison, we have  $W|\delta^W \in L[x]$ , so  $\omega_1^{W[z]} = \omega_1^{L[x]}$ , which implies that  $\delta^W = \omega_1^{L[x]}$ , so  $(W, \emptyset) \notin \mathcal{F}$ . We now proceed to the details.

## 2 The comparison

For a formula  $\varphi$  in the language of set theory (LST),  $\zeta \in \text{OR}$ , and  $x \in \mathbb{R}$ , let  $A_{\varphi, \zeta}^x$  be the set of all  $M \in \text{HC}^{L[x]}$  such that  $L[x] \models \varphi(\zeta, M)$ , and  $L[M]$  is a normally short-tree-iterable pre- $M_1$ -like premouse with  $\delta^{L[M]} = \text{OR}^M$  and  $M = L[M]|\delta^M$ .

Note that  $\varphi$  does not use  $x$  as a parameter. So by absoluteness of normal short-tree-iterability (between  $L[x]$  and  $V$ , for elements of  $\text{HC}^{L[x]}$ ),  $A_{\varphi, \zeta}^x$  is  $\text{OD}^{L[x]}$ . So  $A_{\varphi, \zeta}^x$  is a collection of premice like those involved in the system  $\mathcal{F}$  (restricted to their Woodins).

**Theorem.** *Assume Turing determinacy and that  $M_1^\#$  exists and is fully iterable. Then for a cone of reals  $x$ , for every formula  $\varphi$  in the LST and*

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<sup>2</sup>The point of  $G$  is that we can then use Neeman's genericity iterations, working inside  $L[x]$ . We *cannot* use Woodin's, as closure under Woodin's would produce premice with Woodin cardinal  $\omega_1^{L[x]}$ .

every  $\zeta \in \text{OR}$ , if  $M_1 \restriction \delta^{M_1} \in A_{\varphi, \zeta}^x$  then there is  $R \in A_{\varphi, \zeta}^x$  such that the pseudo-comparison of  $M_1$  with  $L[R]$  has length  $\omega_1^{L[x]}$ .

*Proof.* Suppose not. Then we may fix  $\varphi$  such that for a cone of  $x$ , the theorem fails for  $\varphi, x$ . Fix  $z$  in this cone with  $z \geq_T M_1^\#$ . Let  $\mathcal{W}$  be the  $z$ -genericity iteration on  $M_1$  (making  $z$  generic for the extender algebra), and  $Q = M_\infty^\mathcal{W}$ . By standard arguments (see [7]),  $Q[z] = L[z]$ ,

$$\text{lh}(\mathcal{W}) = \omega_1^{L[z]} + 1 = \delta^Q + 1,$$

$Q \restriction \delta^Q = M(\mathcal{W} \restriction \delta^Q)$ , and  $\mathcal{T} =_{\text{def}} \mathcal{W} \restriction \delta^Q$  is the  $z$ -pseudo-genericity iteration, and  $\mathcal{T} \in L[z]$ .

Let  $\mathbb{B}$  be the extender algebra of  $Q$  and let  $\mathbb{P}$  be the finite support  $\omega$ -fold product of  $\mathbb{B}$ . For  $p \in \mathbb{P}$  let  $p_i$  be the  $i^{\text{th}}$  component of  $p$ . Let  $G \subseteq \mathbb{P}$  be  $Q$ -generic, with  $z_0 = z$  where  $x =_{\text{def}} \langle z_i \rangle_{i < \omega}$  is the generic sequence of reals. Then

$$Q[G] = Q[x] = L[x]$$

and  $x >_T z$ . Let  $\zeta \in \text{OR}$  witness the failure of the theorem with respect to  $\varphi, x$ . So  $M_1 \restriction \delta^{M_1} \in A_{\varphi, \zeta}^x$ .

By [1, Lemma 3.4] (essentially due to Hjorth),  $\mathbb{P}$  is  $\delta^Q$ -cc in  $Q$ , so  $\delta^Q \geq \omega_1^{L[x]}$ , but  $\delta^Q = \omega_1^{L[z]}$ , so  $\delta^Q = \omega_1^{L[x]}$ . So it suffices to see that there is some  $R \in A_{\varphi, \zeta}^x$  such that the pseudo-comparison of  $M_1$  with  $L[R]$  has length  $\delta^Q$ .

For  $e \in \omega$  and  $y \in \mathbb{R}$  let  $\Phi_e^y : \omega \rightarrow \omega$  be the partial function coded by the  $e^{\text{th}}$  Turing program using the oracle  $y$ . Let  $e \in \omega$  be such that  $\Phi_e^z$  is total and codes  $M_1 \restriction \delta^{M_1}$ . Let  $\dot{x}$  be the  $\mathbb{P}$ -name for the  $\mathbb{P}$ -generic sequence of reals, and for  $n < \omega$  let  $\dot{z}_n$  be the  $\mathbb{P}$ -name for the  $n^{\text{th}}$  real. Let  $p \in G$  be such that  $p \Vdash_{\mathbb{P}}^Q \psi(\dot{z}_0)$ , where  $\psi(v)$  asserts “ $\Phi_e^v$  is total and codes a premouse  $R$  such that  $R \in A_{\varphi, \zeta}^{\dot{x}}$ , and the  $v$ -pseudo-genericity iteration of  $L[R]$  produces a maximal tree  $\mathcal{U}$  of length  $\delta^{\check{Q}}$  with  $M(\mathcal{U}) = L[\check{\mathbb{E}}] \restriction \delta^{\check{Q}}$ ”. In the notation of this formula,

$$p \Vdash_{\mathbb{P}}^Q “R \notin \check{V}”, \text{ because } p \Vdash_{\mathbb{P}}^Q “E_0^{\mathcal{U}} \notin M(\mathcal{U})”.$$

By genericity, we may fix  $q \in G$  such that  $q \leq p$  and for some  $m > 0$ ,  $q_m = q_0$ . Note that  $q \Vdash_{\mathbb{P}}^Q \psi(\dot{z}_m)$ .

Let  $\dot{R}_i$  be the  $\mathbb{P}$ -name for the premouse coded by  $\Phi_e^{\dot{z}_i}$  (or for  $\emptyset$  if this does not code a premouse). Also let  $\dot{z}'_0, \dot{z}'_1$  be the  $\mathbb{B} \times \mathbb{B}$ -names for the two  $\mathbb{B} \times \mathbb{B}$ -generic reals (in order), and let  $\dot{R}'_i$  be the  $\mathbb{B} \times \mathbb{B}$ -name for the premouse coded by  $\Phi_e^{\dot{z}'_i}$ .

We may fix  $r \leq q$ ,  $r \in G$ , such that

$$r \Vdash_{\mathbb{P}}^Q \text{“}\dot{R}_0 \neq \dot{R}_m\text{”}. \quad (1)$$

For otherwise there is  $r \leq q$ ,  $r \in G$ , such that  $r \Vdash_{\mathbb{P}}^Q \text{“}\dot{R}_0 = \dot{R}_m\text{”}$ . But since

$$M_1 | \delta^{M_1} = \dot{R}_0^G \notin Q,$$

there are  $s, t \in \mathbb{B}$ ,  $s, t \leq r_0$ , such that

$$(s, t) \Vdash_{\mathbb{B} \times \mathbb{B}}^Q \text{“}\dot{R}'_0 \neq \dot{R}'_1\text{”}.$$

Therefore there are  $u, v \in \mathbb{B}$ , with  $u \leq r_0$  and  $v \leq r_m$ , such that

$$(u, v) \Vdash_{\mathbb{B} \times \mathbb{B}}^Q \text{“}\dot{R}'_0 \neq \dot{R}'_1\text{”}.$$

Let  $w \leq r$  be the condition with  $w_i = r_i$  for  $i \neq 0, m$ , and  $w_0 = u$  and  $w_m = v$ . Then

$$w \Vdash_{\mathbb{P}}^Q \text{“}\dot{R}_0 \neq \dot{R}_m\text{”},$$

a contradiction.

So letting  $R = \dot{R}_m^G$ , we have  $R \neq M_1 | \delta^{M_1}$  and  $R \in A_{\varphi, \zeta}^x$  and  $Q | \delta^Q = M(\mathcal{U})$ , where  $\mathcal{U}$  is the  $z_m^G$ -pseudo-genericity iteration of  $L[R]$ , and  $\text{lh}(\mathcal{U}) = \delta^Q$ . We defined  $\mathcal{T}$  earlier. Let  $\mathcal{T}^*, \mathcal{U}^*$  be the padded trees equivalent to  $\mathcal{T}, \mathcal{U}$ , such that for each  $\alpha$ , either  $E_{\alpha}^{\mathcal{T}^*} \neq \emptyset$  or  $E_{\alpha}^{\mathcal{U}^*} \neq \emptyset$ , and if  $E_{\alpha}^{\mathcal{T}^*} \neq \emptyset \neq E_{\alpha}^{\mathcal{U}^*}$  then  $\text{lh}(E_{\alpha}^{\mathcal{T}^*}) = \text{lh}(E_{\alpha}^{\mathcal{U}^*})$ . Let  $(\mathcal{T}', \mathcal{U}')$  be the pseudo-comparison of  $(M_1, L[R])$ .

We claim that  $(\mathcal{T}', \mathcal{U}') = (\mathcal{T}^*, \mathcal{U}^*)$ ; this completes the proof. For this, we prove by induction on  $\alpha$  that

$$(\mathcal{T}', \mathcal{U}') \restriction (\alpha + 1) = (\mathcal{T}^*, \mathcal{U}^*) \restriction (\alpha + 1).$$

This is immediate if  $\alpha$  is a limit, so suppose it holds for  $\alpha = \beta$ ; we prove it for  $\alpha = \beta + 1$ . Let  $\lambda = \text{lh}(E_{\beta}^{\mathcal{T}^*})$  or  $\lambda = \text{lh}(E_{\beta}^{\mathcal{U}^*})$ , whichever is defined. Because  $M(\mathcal{T}^*) = Q | \delta^Q = M(\mathcal{U}^*)$ , the least disagreement between  $M_{\beta}^{\mathcal{T}^*}$  and  $M_{\beta}^{\mathcal{U}^*}$  has index  $\geq \lambda$ , so we just need to see that  $E_{\beta}^{\mathcal{T}^*} \neq E_{\beta}^{\mathcal{U}^*}$ .

So suppose that  $E_{\beta}^{\mathcal{T}^*} = E_{\beta}^{\mathcal{U}^*}$ . In particular, both are non-empty. Then there is  $s \in G$  such that  $s \leq r$  (see line (1)) and  $s \Vdash_{\mathbb{P}}^Q \sigma$  where  $\sigma$  asserts “For  $i = 0, m$ , let  $\mathcal{T}_i$  be the  $\dot{z}_i$ -pseudo-genericity iteration of  $L[\dot{R}_i]$ . Then  $\mathcal{T}_0$  and  $\mathcal{T}_m$  use identical non-empty extenders  $E$  of index  $\check{\lambda}$ .” Because

$$s \Vdash_{\mathbb{P}}^Q \psi(\dot{z}_0) \ \& \ \psi(\dot{z}_m),$$

also  $s \Vdash_{\mathbb{P}}^Q \sigma'$ , where  $\sigma'$  asserts “Letting  $E$  be as above,  $E \subseteq L[\check{\mathbb{E}}]|\check{\lambda}$ , but  $E \notin \check{V}$ ”; here  $E_{\beta}^{\mathcal{T}^*} \notin Q$  because  $\lambda$  is a cardinal of  $Q$ . But since  $\mathcal{T}_i^G$  is computed in  $Q[z_i^G]$  (for  $i = 0, m$ ) we can argue as before (as in the proof of the existence of  $r$  as in line (1)) to reach a contradiction.  $\square$

A slightly simpler argument, using  $\mathbb{B} \times \mathbb{B}$  instead of  $\mathbb{P}$ , proves the weakening of the theorem given by dropping the parameter  $\zeta$ .

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